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Unsteady fluid and heat flow induced by a submerged stretching surface while its steady motion is slowed down gradually

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Abstract

The title problem arises in the terminal stage of a large class of industrial manufacturing processes as polymer extrusion, wire drawing, drawing of plastic sheets, etc. It concerns the transient crossover to the state of rest of the fluid and heat flow which accompanies the steady fabrication process, when the devices are switched off gradually (i.e. when the motion is slowed down and the surface temperature approaches the ambient temperature continuously). The mechanical and thermal characteristics of such an unsteady process are investigated in the boundary layer approximation, assuming a linear variation of the steady stretching velocity with the longitudinal coordinate x and an inverse linear law for its decrease with time during the gradual switch-off process. For the corresponding surface temperature a general power-law variation is admitted. The paper presents the similarity analysis of several specific cases. The cases of basic interest of a constant surface temperature T_w and of a constant surface heat flux q_w are discussed in some detail. In the case $T_w = \text{const.}$ an exact solution is reported and the Prandtl number dependence of the corresponding surface heat flux is given for all $0 < Pr < \infty$.

Keywords: Linear stretching; Boundary layer; Similarity solution; Isothermal surface; Isoflux condition

1. Introduction

Fluid and heat flow induced by continuous stretching surfaces submerged in a quiescent fluid medium has many engineering applications. Such applications are encountered in metal and polymer extrusion, continuous casting, drawing of plastic films, wire drawing, etc. In the present investigation we are interested in modeling the unsteady two-dimensional boundary layer flow and heat transfer induced by a surface after its steady linear speed is slowed down gradually. Accordingly, only the relevant previous works related to linearly stretching surfaces will be quoted here; for a comprehensive list of general references on flow and heat transfer induced by moving surfaces the reader is referred to Ali and Al-Yousef [1].

The steady two-dimensional flow and heat transfer was analyzed by Crane [2] for linearly moving impermeable sheet with uniform surface temperature. Gupta and Gupta [3], Grubka and Bobba [4], Chen and Char [5], Ali [6,7] and Magyari et al. [8] have studied the same case for permeable surfaces and different surface temperature distributions. Furthermore, mixed convection heat transfer from a linearly moving surface was reported by Ali and Al-Yoysef [1] and by Chen [9]. A new approach which applies the Merkin transformation method to the heat transfer problems of steady boundary layer flows induced by stretching surfaces has recently been reported by Magyari and Keller [10].

Similarity solutions of the unsteady Navier–Stokes equations, of a thin liquid film on a stretching sheet were

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Nomenclature

c	wall temperature exponent (t-variation)	Λ	dimensionless constant, $\Lambda = \gamma L/u_0$
f	similarity dependent variable	μ	dynamic viscosity
k	thermal conductivity	υ	kinematic viscosity, $\upsilon = \mu/\rho$
L	reference length	η	similarity independent variable
n	wall temperature exponent (x-variation)	ρ	density
Pr	Prandtl number, $Pr = \upsilon/\alpha$	θ	dimensionless temperature
Re	Reynolds number, $Re = u_0L/\upsilon$	τ	dimensionless time, $\tau = \gamma t$
q	heat flux	τ_w	wall shear stress
S	dimensionless wall shear stress	ψ	dimensionless stream function
t	time variable		
T	temperature	*Subscripts*	
u	longitudinal velocity component	w	wall condition
U	dimensionless longitudinal velocity	∞	condition at infinity
v	transversal velocity component	0	reference value
V	dimensionless transversal velocity	x, y, τ	partial derivatives
x, y	Cartesian coordinates		
X, Y	dimensionless Cartesian coordinates	*Superscript*	
		s	steady state
Greek symbols			
α	thermal diffusivity		
γ	unsteadiness parameter		

considered by Wang [11] and by Usha and Sridharan [12] for the axisymmetric case. The same problem was extended by Andersson et al. [13] to fluids obeying non-Newtonian constitutive equations. The fluid velocity and skin friction coefficient for an unsteady flow past a wall which starts to move impulsively from the rest, have been calculated by Pop and Na [14], using both the series and numerical solution methods. Furthermore, the heat transfer characteristics of the flow problem of Wang [11] was considered by Andersson et al. [15]. The effect of the unsteadiness parameter on heat transfer and flow field over a stretching surface with and without heat generation was considered by Elbashbeshy and Bazid [16,17], respectively.

2. Basic equations

The analysis starts with the continuity, momentum and thermal energy equations

$$u_x + v_y = 0 \tag{1}$$

$$u_t + uu_x + vu_y = \upsilon u_{yy} \tag{2}$$

$$T_t + uT_x + vT_y = \alpha T_{yy} \tag{3}$$

The subscripts denote partial derivatives with respect to x, y, t.

We assume that for $t < 0$ the fluid and heat flow are steady, i.e.

$$u = u^s(x,y), \quad v = v^s(x,y), \quad T = T^s(x,y) \tag{4}$$

where (u^s, v^s, T^s) is the steady state solution of Eqs. (1)–(3) satisfying the boundary conditions

$$\left.\begin{array}{l} u^s \equiv u^s_w(x) = u_0\frac{x}{L}, \quad v^s \equiv v^s_w(x), \\ T^s \equiv T^s_w(x) = T_\infty + T_0\left(\frac{x}{L}\right)^n \end{array}\right\} \text{ on } y = 0 \tag{5}$$

$$u^s \to 0, \quad T^s \to T_\infty \quad \text{as } y \to \infty \tag{6}$$

The unsteady fluid and heat flow starts at $t = 0$ with the initial conditions

$$u(x,y,t) = u^s(x,y), \quad v(x,y,t) = v^s(x,y),$$
$$T(x,y,t) = T^s(x,y) \quad \text{at } t = 0 \tag{7}$$

such that the unsteady state evolves for $t > 0$ according to the full balance equations (1)–(3) under the boundary conditions

$$\left.\begin{array}{l} u \equiv u_w(x,t) = \frac{u^s_w(x)}{1+\gamma t}, \quad v \equiv v_w(x,t), \\ T \equiv T_w(x,t) = T_\infty + \frac{T_0}{(1+\gamma t)^c}\left(\frac{x}{L}\right)^n \end{array}\right\} \text{ on } y = 0 \tag{8}$$

$$u \to 0, \quad T \to T_\infty \quad \text{as } y \to \infty \tag{9}$$

In the above equations the constants u_0, T_0, T_∞ and γ are positive, c and n are arbitrary, and L is some reference length which will be specified below.

3. Non-dimensionalization

The present unsteady stretching problem possesses a characteristic velocity scale u_0 and a characteristic time scale γ^{-1}. We non-dimensionalize the mass and momentum balance equations according to

$$X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad \tau = \gamma t$$
$$U(X, Y, \tau) = \frac{u(x, y, t)}{u_0}, \quad V(X, Y, \tau) = \frac{v(x, y, t)}{u_0} \tag{10}$$

and obtain

$$U_X + V_Y = 0 \tag{11}$$

$$\Lambda \cdot U_\tau + UU_X + VU_Y = \frac{1}{Re} U_{YY} \tag{12}$$

where

$$Re = \frac{u_0 L}{\upsilon} \tag{13}$$

is the Reynolds number and

$$\Lambda = \frac{\gamma L}{u_0} \tag{14}$$

is a further positive dimensionless constant.

We now introduce the dimensionless stream function ψ by the definition

$$U = \psi_Y, \quad V = -\psi_X \tag{15}$$

Thus we are left with the single equation for the fluid flow,

$$\Lambda \cdot \psi_{\tau Y} + \psi_Y \psi_{XY} - \psi_X \psi_{YY} = \frac{1}{Re} \psi_{YYY} \tag{16}$$

The thermal energy equation becomes

$$\Lambda \cdot T_\tau + \psi_Y T_X - \psi_X T_Y = \frac{1}{Pr \cdot Re} T_{YY} \tag{17}$$

where $Pr = \upsilon/\alpha$ is the Prandtl number.

4. Similarity transformation

We introduce the similarity transformation

$$\psi = \frac{X}{\sqrt{Re}} (1+\tau)^{-1/2} f(\eta)$$
$$\eta = \sqrt{Re} \cdot (1+\tau)^{-1/2} \cdot Y \tag{18}$$
$$T = T_\infty + T_0 X^n (1+\tau)^{-c} \theta(\eta)$$

where T_0 is a (positive or negative; heating or cooling) reference temperature.

The components of the dimensionless velocity field result as

$$U(X, Y, \tau) = X(1+\tau)^{-1} f'(\eta)$$
$$V(X, Y, \tau) = -\frac{1}{\sqrt{Re}} (1+\tau)^{-1/2} f(\eta) \tag{19}$$

Their dimensional counterparts are given by

$$u(x, y, t) = u_0 X (1+\tau)^{-1} f'(\eta)$$
$$v(x, y, t) = -\frac{u_0}{\sqrt{Re}} (1+\tau)^{-1/2} f(\eta) \tag{20}$$

For the similar stream function f and the similar temperature θ the following equation emerge:

$$f''' + ff'' - f'^2 + \Lambda \cdot \left(f' + \frac{1}{2}\eta f''\right) = 0 \tag{21}$$

$$\frac{\theta''}{Pr} + f\theta' - nf'\theta + \Lambda \cdot \left(c\theta + \frac{1}{2}\eta\theta'\right) = 0 \tag{22}$$

The dimensional boundary conditions (8) and (9) become

$$\left.\begin{aligned} u &\equiv u_w(x,t) = u_0 X (1+\tau)^{-1} f'(0) \\ v &\equiv v_w(t) = -\frac{u_0}{\sqrt{Re}} (1+\tau)^{-1/2} f(0) \end{aligned}\right\} \quad (\text{on } y = 0) \tag{23}$$

$$T \equiv T_w(x,t) = T_\infty + T_0 X^n (1+\tau)^{-c} \theta(0) \quad (\text{on } y = 0) \tag{24}$$

and together with (9) imply for the dimensionless functions f and θ the conditions

$$f(0) = f_w, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{25}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0 \tag{26}$$

In addition to the boundary conditions (25) and (26) the requirements

$$f'(\eta) \geqslant 0 \quad \text{for all } \eta \geqslant 0 \tag{27}$$

and

$$\theta(\eta) \geqslant 0 \quad \text{for all } \eta \geqslant 0 \tag{28}$$

must also be satisfied. The former condition aims to avoid the flow reversal (where the boundary layer approximation breaks down) and the latter one excludes the violation of the first principle of thermodynamics.

5. General features of the basic boundary value problems

5.1. Scaling behavior

Our basic boundary value problem is specified by Eqs. (21), (22) and (25)–(28). It splits in an independent flow boundary value problem (21), (25) and (27) and in the forced thermal convection problem (22), (26) and (28), respectively.

As mentioned in Section 3, the present unsteady stretching problem possesses a natural velocity scale u_0 and a natural time scale γ^{-1}. These specify in turn also the natural length scale $L_{nat} = u_0 \cdot \gamma^{-1}$. The flow may obviously be observed also on any arbitrarily chosen length scale L. The parameter Λ defined by Eq. (14) represents precisely the ratio of such an arbitrary length scale L and the natural one L_{nat}. As a consequence we may choose in the above equations $\Lambda = 1$ without any lost of (physical) generality since $\Lambda \neq 1$ corresponds to one and the same physical flow observed on other length scales than the natural one $L_{nat} = u_0 \cdot \gamma^{-1}$. In this sense the steady problem ($\gamma = 0$) which corresponds in Eqs. (21) and (22) to the value $\Lambda = 0$ can also be interpreted as the unsteady problem ($\gamma \neq 0$) observed on the length scale $L = 0$. When not specified otherwise, we chose hereafter $L = L_{nat} = u_0 \cdot \gamma^{-1}$, i.e. $\Lambda = 1$.

In addition to the velocity and temperature fields, further quantities of engineering interest are the wall shear stress $\tau_w(x,t) = \rho\upsilon\partial u/\partial y|_{y=0}$ and wall heat flux $q_w(x,t) = -k\partial T/\partial y|_{y=0}$ which in the present case become

$$\tau_w(x,t) = \frac{\rho u_0^2}{\sqrt{Re}} X(1+\tau)^{-3/2} f''(0) \tag{29}$$

$$q_w(x,t) = -\frac{kT_0}{L}\sqrt{Re}\cdot X^n (1+\tau)^{-c-1/2}\theta'(0) \tag{30}$$

5.2. Integral relationships

Assuming a sufficiently rapid decay of $f'(\eta)$ and $\theta(\eta)$ as well as a finite value $f(\infty) = f_\infty$ of the similar entrainment velocity, one easily obtains the integral relationships

$$f''(0) = \frac{1}{2}(f_\infty - 3f_w) - 2\int_0^\infty f'^2 \mathrm{d}\eta \tag{31}$$

$$\theta'(0) = -Pr\left[f_w + (n+1)\int_0^\infty f'\theta \mathrm{d}\eta - \left(c - \frac{1}{2}\right)\int_0^\infty \theta \mathrm{d}\eta\right] \tag{32}$$

Eq. (31) yields the inequality

$$f''(0) \equiv S(f_w) < \frac{1}{2}(f_\infty - 3f_w) \tag{33}$$

5.3. The Reynolds analogy

If $f = f(\eta)$ solves the flow problem for a specified value of f_w, then

$$\theta = f'(\eta) \tag{34}$$

solves the temperature problem for

$$Pr = 1, \quad n = 1, \quad c = 1 \tag{35}$$

This is the Reynolds analogy applied to the present problem. Hence, in this case we have

$$\theta'(0) = f''(0) \equiv S(f_w) \tag{36}$$

6. Solutions of the flow problem

The flow problem (21) and (25) can easily be solved numerically, e.g. with the aid of the shooting method. In Fig. 1 the plots of $f(\eta)$ and $f'(\eta)$ of a possible solution for an impermeable surface ($f_w = 0$) are shown. It is found that $S(0) = -0.651216$ is only the smallest value of $S(0)$ for which the flow problem for an impermeable surface admits solution satisfying the additional condition (27). In fact, for $f_w = 0$ it admits such a solution for any value of $f''(0) \equiv S(0)$ in the range

$$S_{\min}(0) = -0.651216 \leqslant S(0) < 0 \tag{37}$$

In other words, for $f_w = 0$ the flow problem admits a family of multiple solutions: one solution for every value of $S(0)$ in the interval (37). This feature of the problem (21) and (25) is illustrated in Fig. 2. It is interesting to notice that the dimensionless entrainment velocity f_∞ is finite only for the solution corresponding to $S(0) = S_{\min}(0) = -0.651216$ and it becomes infinite for all the other members of the family of solutions with $S_{\min}(0) < S(0) < 0$. This circumstance is illustrated in Fig. 3. As a consequence, only the (rapidly decaying) velocity profile associated with $S(0) = S_{\min}(0) = -0.651216$ corresponds to a physical solution of our flow problem for $f_w = 0$, the other ones with $S_{\min}(0) < S(0) < 0$ are non-physical. Furthermore, for $S(0) > 0$ no solutions exist and for $S(0) < S_{\min}(0)$ the solutions of the problem (21) and (25) violate the condition (27), i.e. ranges of negative values of $f'(\eta)$ exist for $S(0) < S_{\min}(0)$. This property is illustrated in Fig. 4. Concerning the asymptotic behavior of the similar velocity profiles plotted in Figs. 1–4, it is worth mentioning here that in addition to the boundary condition

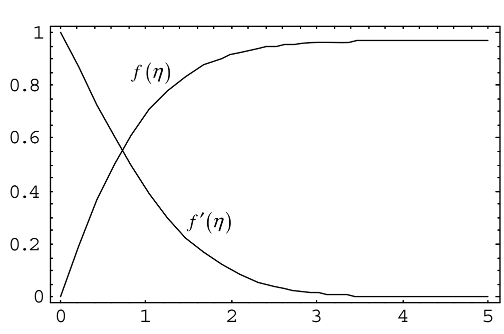

Fig. 1. Plots of $f(\eta)$ and $f'(\eta)$ for $f_w = 0$. In this case $f(\infty) = 0.969014$ and $f''(0) \equiv S(0) = -0.651216$.

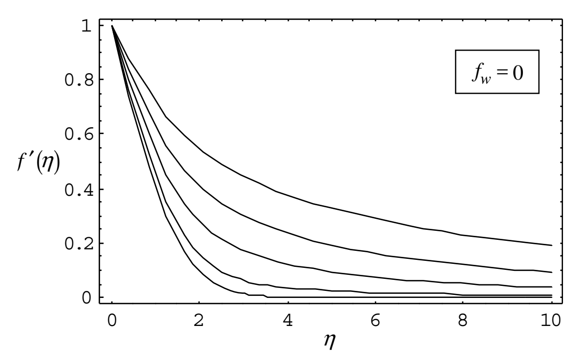

Fig. 2. Dimensionless velocity profiles $f'(\eta)$ associated with $f_w = 0$ and five different values of $S(0)$ in the interval (37), $S(0) = -0.3, -0.4, -0.5, -0.6$ and -0.651216 (from top to bottom) where no backflow exists.

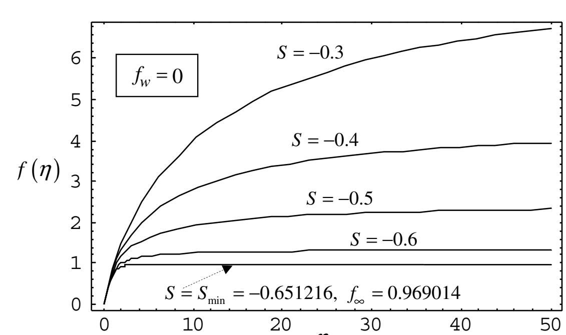

Fig. 3. Profiles of $f(\eta)$ associated with $f_w = 0$ and five different values of $S(0)$ in the interval (37) where no backflow exists.

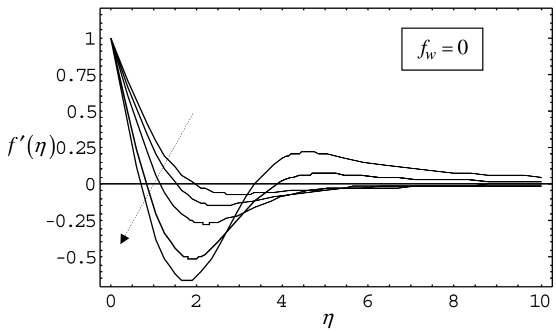

Fig. 4. Dimensionless velocity profiles $f'(\eta)$ associated with $f_w = 0$ and five different values of $S(0)$ in the range $S(0) < S_{min}(0) = -0.651216$, namely $S(0) = -0.75, -0.85, -1.3$ and -1.5 (from top to bottom, as indicated by the arrow). These solutions violate the condition (27), i.e. ranges of negative $f'(\eta)$ exist.

$f'(\infty) = 0$, the physical solutions $f'(\eta)$ must also satisfy the condition $f'(\infty) = 0$ of a smooth asymptotic decay.

When the surface is permeable and a lateral suction ($f_w > 0$) or injection ($f_w < 0$) of the fluid is applied, the boundary value problem admits one solution without flow reversal for every point of coordinate (S,f_w) belonging to the second or the and third quadrant of the parameter plane (S,f_w) assuming that, for a specified value of the suction/injection parameter f_w, the point lies either (i) on the border curve $S = S_{min}(f_w)$, or (ii) above it, i.e.

$$S_{min}(f_w) \leqslant S < 0 \tag{38}$$

This domain of existence of the solutions without flow reversal is represented by the grey region of the parameter plane shown in Fig. 5. In Figs. 6 and 7 the solutions $f'(\eta)$ and $f(\eta)$ corresponding to three different points of the border curve are shown. Similarly to the case $f_w = 0$ illustrated in Fig. 3, the dimensionless entrainment velocity f_∞ is finite also for $f_w \neq 0$ only for the solutions corresponding to the points of the border curve $S = S_{min}(f_w)$ and it becomes infinite for all the other points of the existence domain shown in Fig. 5.

We may conclude, therefore, that only the (rapidly decaying) velocity profiles associated with the points of the border curve $S = S_{min}(f_w)$ correspond to physical solutions of our flow boundary value problem (21) and (25). They also satisfy the additional condition (27) of non-existence of regions with flow reversal.

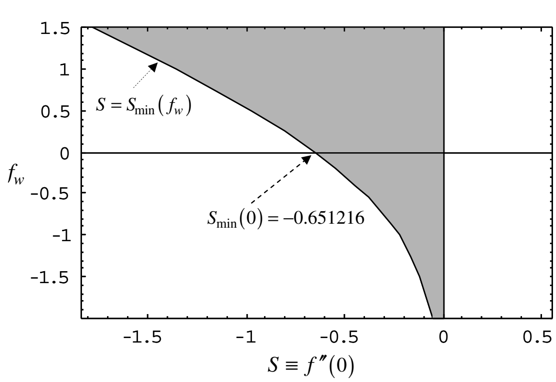

Fig. 5. Existence domain of solutions of the flow boundary value problem (21) and (25) satisfying the additional condition (27) (the grey region of the parameter plane (S,f_w)).

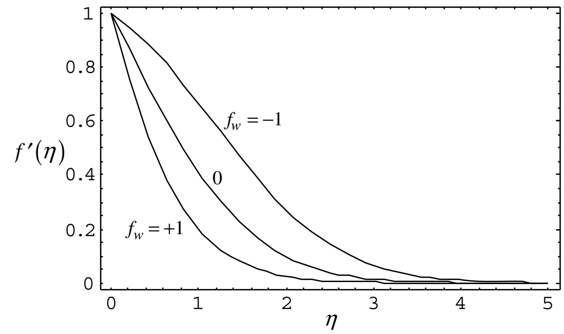

Fig. 6. Solutions $f'(\eta)$ of the flow problem corresponding to three different points of the border curve shown in Fig. 5. The coordinates of these points in the parameter plane (S,f_w) are: (S,f_w) = (−1.3551, +1), (−0.6512, 0), and (−0.2235, −1), respectively.

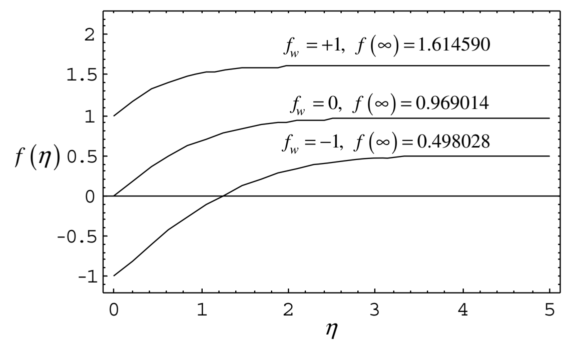

Fig. 7. Solutions $f(\eta)$ of the flow problem corresponding to three different points of the border curve shown in Fig. 5. The coordinates of these points in the parameter plane (S,f_w) are the same as in Fig. 6.

7. Solutions of the temperature problem

We investigate in this section different physically relevant cases of the temperature boundary value problem (22) and (26) for $\Lambda = 1$ along with the additional condition (28).

The Reynolds-analogy discussed in Section 5.3 yields already a class of such solutions in terms of solutions of the flow problem presented in Section 6.

7.1. The special case $n = -1$, $c = +1/2$

In this case the integral relationship (32) yields for the dimensionless surface temperature gradient the explicit result

$$\theta'(0) = -Prf_w \quad (n = -1,\ c = 1/2) \tag{39}$$

Hence, for a given value of the suction/injection parameter f_w the wall heat flux scales linearly with Pr.

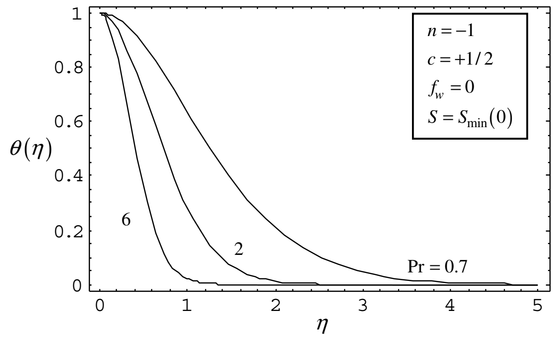

Fig. 8. Plots of $\theta(\eta)$ given by Eq. (41) for $f_w = 0$, $f''(0) = S_{min}(0) = -0.651216$ and three different values of the Prandtl number. In all of these cases the similar wall heat flux is zero, $\theta'(0) = 0$.

The thermal boundary value problem (22) and (26) can now be transcribed into the form

$$\frac{d}{d\eta}\left[\frac{\theta'}{Pr} + \left(f + \frac{1}{2}\eta\right)\theta\right] = 0 \qquad \theta(0) = 1, \quad \theta(\infty) = 0 \tag{40}$$

The solution of (40) can be obtained by quadratures and reads

$$\theta(\eta) = e^{-Pr \cdot F(\eta)} \tag{41}$$

where the notation

$$F(\eta) \equiv \frac{1}{4}\eta^2 + \int_0^\eta f(\tilde{\eta})d\tilde{\eta} = \int_0^\eta \left[\frac{1}{2}\tilde{\eta} + f(\tilde{\eta})\right]d\tilde{\eta} \tag{42}$$

has been used.

The dimensional temperature field and the wall heat flux are given in this case by

$$T = T_\infty + T_0 \frac{L}{x}\frac{1}{\sqrt{1+\tau}}\theta(\eta) \tag{43}$$

and

$$q_w(x,t) = \frac{kT_0}{L} f_w \cdot \sqrt{Re} \cdot \frac{L}{x}\frac{Pr}{1+\tau} \tag{44}$$

respectively, where as explained in Section 5.1, $L = L_{nat} = u_0 \cdot \gamma^{-1}$.

In the case $f_w = 0$, Eq. (43) describes the temperature field over an insulated impermeable surface. Indeed, according to Eq. (44) the heat flux is vanishing for $f_w = 0$ in every point of the surface except for the leading edge singularity at $x = 0$. The heat responsible for the temperature field is released by this singularity.

In Fig. 8 the temperature profiles (41) are shown for three different values of the Prandtl number and the flow solution corresponding to $f_w = 0$ and $f''(0) = S_{min}(0) = -0.651216$ (see Section 6).

7.2. Prescribed constant wall temperature: T_w = const.

This is the important case of an isothermal stretching surface corresponding to the temperature exponents

$$n = 0, \quad c = 0 \tag{45}$$

which yields the temperature boundary value problem

$$\frac{\theta''}{Pr} + \left(f + \frac{1}{2}\eta\right)\theta' = 0 \qquad \theta(0) = 1, \quad \theta(\infty) = 0 \tag{46}$$

The corresponding dimensional temperature filed and wall heat flux are

$$T = T_\infty + (T_w - T_\infty)\theta(\eta) \tag{47}$$

and

$$q_w = -\frac{k}{L}(T_w - T_\infty)\sqrt{\frac{Re}{1+\tau}} \cdot \theta'(0) \tag{48}$$

respectively. The quantity of engineering interest in this case is the similar wall temperature gradient $\theta'(0)$ as a function of Pr.

The solution of the boundary value problem (46) can again be given by quadratures,

$$\theta(\eta) = 1 + \theta'(0) \cdot \int_0^\eta e^{-Pr \cdot F(\bar{\eta})} d\bar{\eta} \tag{49}$$

where $F(\eta)$ is given by Eq. (42) and the similar wall temperature gradient $\theta'(0)$ by equation

$$\theta'(0) = -\left(\int_0^\infty e^{-Pr \cdot F(\bar{\eta})} d\bar{\eta}\right)^{-1} \tag{50}$$

In Fig. 9 the temperature profiles (49) are shown for three different values of the Prandtl number and the flow solution corresponding to $f_w = 0$ and $f''(0) = S_{min}(0) = -0.651216$ (see Section 6). The dependence of the similar wall temperature gradient on the Prandtl number is shown in Fig. 10a where $-\theta'(0)$ has been plotted versus Pr according to Eq. (50) for $f_w = 0$.

With the aid of Eq. (50) can be shown that $-\theta'(0)$ scales (for $f_w = 0$ and $f''(0) = S_{min}(0) = -0.651216$) with $\sqrt{Pr}$ both for small and large values of the Pr, namely

$$\theta'(0) = \begin{cases} -0.60264 \cdot \sqrt{Pr} & \text{as } Pr \to 0 \\ -0.96782 \cdot \sqrt{Pr} & \text{as } Pr \to \infty \end{cases} \quad (f_w = 0,\ S = S_{min}(0)) \tag{51}$$

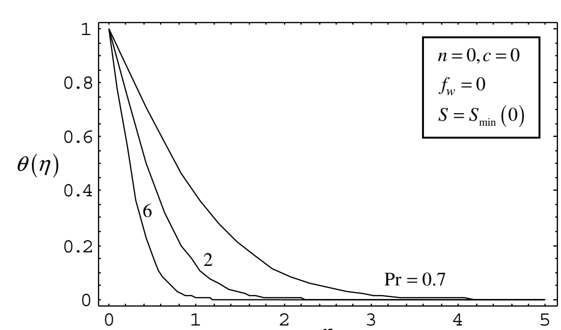

Fig. 9. Plots of $\theta(\eta)$ for T_w = const., $f_w = 0$, $f''(0) = S_{min}(0) = -0.651216$ and the values $Pr = 0.7$, 2 and 6 of the Prandtl number. The corresponding values of the wall temperature gradient are $\theta'(0) = -0.714448$, -1.279805 and -2.294967, respectively.

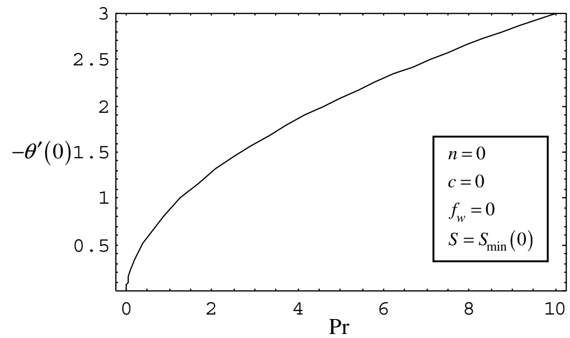

Fig. 10a. Plot of $-\theta'(0)$ as a function of Pr in the case $T_w = \text{const.}$ for $f_w = 0$, and $S = S_{min}(0) = -0.651216$.

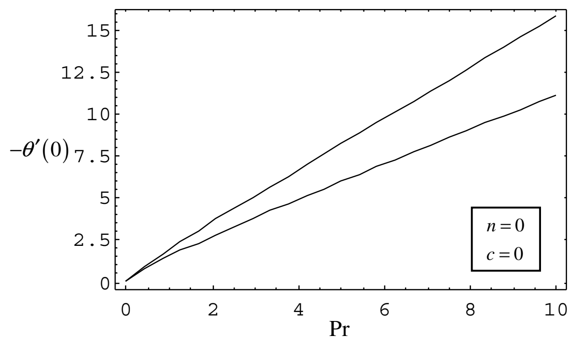

Fig. 10b. Plot of $-\theta'(0)$ as a function of Pr in the case $T_w = \text{const.}$ for $f_w = +1$, $S = S_{min}(+1) = -1.3551$ (lower curve) and $f_w = +1.5$, $S = S_{min}(+1.5) = -1.77673$ (upper curve), respectively.

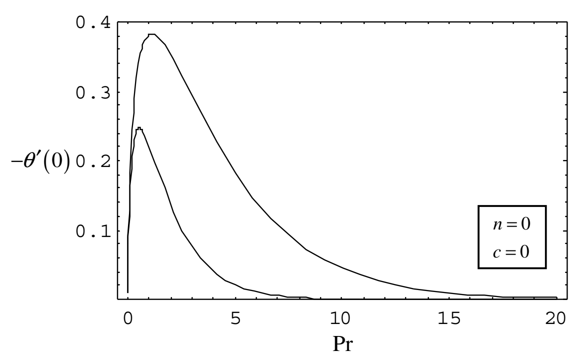

Fig. 10c. Plot of $-\theta'(0)$ as a function of Pr in the case $T_w = \text{const.}$ for $f_w = -1$, $S = S_{min}(-1) = -0.22351$, (upper curve) and $f_w = -1.5$, $S = S_{min}(-1.5) = -0.113072$, (lower curve), respectively.

We mention that this unusual correlation (the low-and high-Pr asymptotics exhibit the same Pr-dependence), although highly accurate, is an approximate scaling relationship since the numerical coefficient of $\sqrt{Pr}$ in Eq. (51) also shows a slight dependence on Pr. However, the small increase from 0.60264 to 0.96782 takes place while the Prandtl number varies over 6 orders of magnitude.

Compared with the case (51) of the impermeable surface ($f_w = 0$), in the permeable case the lateral suction ($f_w > 0$) or injection ($f_w < 0$) of the fluid has an essential effect on the Prandtl number dependence of the surface temperature

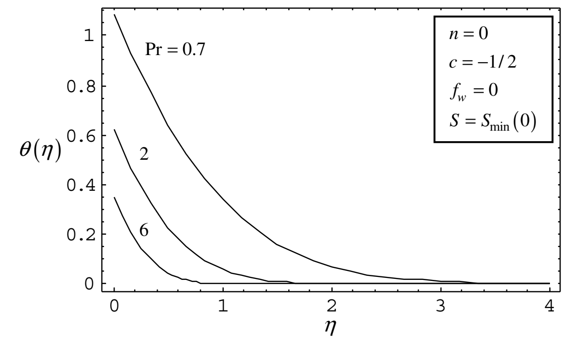

Fig. 11. Plots of $\theta(\eta)$ for $q_w = \text{const.}$, $f_w = 0$, $f''(0) = S_{min}(0) = -0.651216$ and the values $Pr = 0.7, 2$ and 6 of the Prandtl number. The corresponding values of the similar wall temperature are $\theta(0) = 1.084548$, 0.622409 and 0.352565, respectively.

gradient $\theta'(0)$. This effect is illustrated in Figs. 10b and 10c for ($f_w = +1$, $S = S_{min}(+1) = -1.3551$), ($f_w = +1.5$, $S = S_{min}(+1.5) = -1.77673$), ($f_w = -1$, $S = S_{min}(-1) = -0.22351$) and ($f_w = -1.5$, $S = S_{min}(-1.5) = -0.113072$), respectively.

7.3. Prescribed constant wall heat flux: q_w = const.

According to Eq. (30) a constant prescribed value of the wall heat flux corresponds to the temperature exponents

$$n = 0, \quad c = -1/2 \tag{52}$$

The quantity of engineering interest in this case is the similar wall temperature $\theta(0)$ as a function of Pr while the value of the similar wall temperature gradient $\theta'(0)$ may be specified arbitrarily, the usual choice being $\theta'(0) = -1$. Thus the corresponding thermal boundary value problem reads

$$\begin{aligned} &\frac{\theta''}{Pr} + \left(f + \frac{1}{2}\eta\right)\theta' - \frac{1}{2}\theta = 0 \\ &\theta'(0) = 1, \quad \theta(\infty) = 0 \end{aligned} \tag{53}$$

The corresponding dimensional temperature field is

$$T = T_\infty + \frac{Lq_w}{k}\sqrt{\frac{1+\tau}{Re}} \cdot \theta(\eta) \tag{54}$$

The surface temperature becomes in this case

$$T_w = T_\infty + \frac{Lq_w}{k}\sqrt{\frac{1+\tau}{Re}} \cdot \theta(0) \tag{55}$$

In Fig. 11 the temperature profiles $\theta(\eta)$ as solutions of the boundary value problem (53) are shown for $f_w = 0$, $f''(0) = S_{min}(0) = -0.651216$ and three different values of the Prandtl number.

8. Summary and conclusions

The unsteady heat and fluid flow which occurs during the continuous switch-off of some industrial manufacturing processes, as e.g. the drawing of plastic sheets, has been investigated in this paper on a model system. A linear variation of the steady stretching velocity with the longitudinal coordinate x and an inverse linear law for its decrease with time during the gradual switch-off process have been assumed. For the corresponding surface temperature a general power-law variation is admitted. A similarity analysis of several special cases has been presented.

The main results of the paper can be summarized as follows:

1. For any specified value of the suction injection parameter f_w in the range $-\infty < f_w < +\infty$ the flow problem admits solutions without backflow regions only if the surface shear stress $f''(0) \equiv S(f_w)$ equals or exceeds a minimum value $S_{min}(f_w)$. These values specify in the parameter plane (f_w, S) a "border curve" which is shown in Fig. 5. The physical solutions of the flow problem associated with finite values of the entrainment velocity correspond to the points of the border curve. From each point of this curve there bifurcates a family of further solutions characterized by slowly decaying velocity profiles and infinite entrainment velocities.
2. For constant surface temperature T_w, an exact solution has been given for the similar temperature which in turn allows to calculate the corresponding surface heat flux for any value of the Prandtl number (see Fig. 10a). There has been found that, as expected, the lateral suction or injection of the fluid exerts a strong effect on the surface heat transfer (see Figs. 10b and 10c).
3. For constant surface heat flux q_w, numerical solutions have been presented for the similar temperature profiles $\theta(\eta)$ in the case of an impermeable surface ($f_w = 0$). A comprehensive study of the effect of suction and injection in this case is still open. The investigation of the Prandtl number-dependence of the similar surface temperature $\theta(0)$ for $q_w = \text{const.}$ is a further research opportunity.
4. An analysis of the involved physical scales has shown (see Section 5.1) that the parameter Λ which occurs in the basic equations (21) and (22) of the problem, may be chosen without any lost of physical generality equal to the unity, since $\Lambda \neq 1$ corresponds to one and the same physical flow observed on different length scales. In other words, a parameter study of the basic boundary value problem with respect to Λ (as it is encountered in several earlier publications) yields no new physics, but it is only of a mathematical interest.

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